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Vector coherent state theory of the semidirect sum Lie algebras wsp($2N$, $\mathbb{R}$)

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Abstract. Positive discrete series irreducible representations of the semidirect sum Lie algebras wsp($2N,\mathbb{R}$) = w(N) $\oplus$ sp($2N,\mathbb{R}$) are given in wsp($2N,\mathbb{R}$) $\supset$ sp($2N,\mathbb{R}$) $\supset$ u(N) bases. The branching rule for the reduction of a wsp($2N,\mathbb{R}$) irreducible representation into a direct sum of multiplicity-free sp($2N,\mathbb{R}$) irreducible representations is accounted for by the explicit construction of a wsp($2N,\mathbb{R}$) $\supset$ sp($2N,\mathbb{R}$) $\supset$ u(N) basis of vector-valued holomorphic functions. Analytic results are obtained for the matrix elements of the wsp($2N,\mathbb{R}$) mod sp($2N,\mathbb{R}$) generators between basis states of lowest-weight u(N) irreducible representations. The wsp($2N,\mathbb{R}$) example shows that the applicability of the vector coherent state and K-matrix combined theory is not restricted to semisimple (and, more generally, reductive) Lie algebras, therefore pointing to a further extension of the method. In addition, it emphasises the importance of the graded structure of Lie algebras with respect to some u(1) generator belonging to their stability subalgebra.

1. Introduction

During the last few years, the vector coherent state (VCS) and K-matrix combined theory has played an ever increasing role in Lie group and Lie algebra representation theory (for a list of references see, for example, Hecht (1987), Rowe *et al* (1988)). Under some rather mild conditions to be detailed below, it indeed provides a simple systematic procedure for determining the explicit matrices for the ladder irreducible representations (irreps) of Lie groups and Lie algebras.

VCS (also called partially coherent states) were introduced (Rowe 1984, Rowe *et al* 1985a, Deenen and Quesne 1984b, 1985, Quesne 1986a, b) as a natural extension of standard (generalised) coherent states (Perelomov 1972, 1977). Such an extension is directly related to Mackey's induced representation theory (Mackey 1968). Instead of inducing the ladder irrep of a group G from one-dimensional irreps of a subgroup H, as in standard coherent state theory, in its vector generalisation the induction is performed from finite-dimensional vector representations of H. The group G is then represented as a group of linear transformations in a Hilbert space of holomorphic vector-valued functions of a set of complex variables.

The corresponding Lie algebra g is realised as a subalgebra of the universal enveloping algebra of some contracted Lie algebra $\bar{\mathrm{g}}$. The latter is the direct sum of a so-called 'intrinsic' representation of the subalgebra h associated with H and of a

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Heisenberg-Weyl algebra $\mathrm{w}(\nu)$ for ν independent sets of boson creation and annihilation operators. Such a construction of g by expansion of $\bar{\mathrm{g}}$ is known by physicists as a boson representation of g (Deenen and Quesne 1982, 1985). Hence VCS theory is also related to Lie algebra boson representations.

The integral form of the VCS identity resolution being both difficult to obtain and cumbersome to use, an alternative procedure was developed for determining the VCS scalar product. This is the K-matrix technique (Rowe 1984, Rowe *et al* 1984, 1988, Deenen and Quesne 1984c, 1985, Castaños *et al* 1985, 1986, Hecht 1987, Le Blanc and Rowe 1988), which originated from a method (Deenen and Quesne 1982) used to convert a Lie algebra Dyson boson representation (Dyson 1956) into a Holstein-Primakoff representation (Holstein and Primakoff 1940). In this approach, the VCS scalar product is defined by specifying an orthonormal basis. For such purpose, a basis orthonormal with respect to a Bargmann scalar product (Bargmann 1961) is first constructed, then mapped onto an orthonormal VCS basis by means of a transformation K. In both steps, the full power of tensor calculus is exploited by making use of irreducible tensor coupling and Wigner-Eckart theorem with respect to h. Since the square of K obeys a linear recursion relation, it can be determined either numerically or even in some cases analytically. Once the transformation K has been obtained, the explicit matrices of the g generators in the orthonormal basis can be easily inferred.

The applicability of the VCS and K-matrix combined theory is restricted to those cases where the subalgebra h of g is a compact subalgebra of maximal rank and g has a $(2s+1)$-dimensional graded structure with respect to some generator $\mathcal{N}$ of h, for which $[\mathcal{N}, \mathrm{h}]=0$. Such Lie algebra gradings play a basic role in the Tits-Koecher construction of finite-dimensional simple Lie algebras (Tits 1962, Koecher 1967), and in its extension by Kantor (1972). Since the building blocks of the latter are the ternary algebras (Bars and Günaydin 1979), VCS theory is also related to these fundamental mathematical structures.

K-matrix theory was originally introduced and applied to Lie algebras admitting a three-dimensional graded structure or Jordan decomposition (Rowe 1984, Rowe *et al* 1984, Deenen and Quesne 1984c, 1985, Castaños *et al* 1985, 1986, Hecht 1987). Recently, it was generalised to also cover the Lie algebras with a five-dimensional graded structure or Kantor decomposition (Rowe *et al* 1988). Moreover, by studying an example of Lie algebra with a seven-dimensional graded structure, it was conjectured (Le Blanc and Rowe 1988) that this extended K-matrix theory is in fact valid for all $(2s+1)$-dimensional gradings, whatever the value of s may be (except for the $s=1$ case, for which drastic simplifications occur).

In both the theoretical developments and the applications so far carried out, only semisimple (or, more generally, reductive) Lie algebras were considered. However, as a close inspection would disclose, neither the VCS construction nor the K-matrix technique applicability do in fact depend in any essential way on the semisimplicity of g. This point will be illustrated in the present paper by considering the case of the positive discrete series irreps of the semidirect sum Lie algebras $\mathrm{wsp}(2N,\mathbb{R}) = \mathrm{w}(N) \oplus\!\!\!\!\!\raisebox{0.3ex}{$\cdot$}\; \mathrm{sp}(2N,\mathbb{R})$ in $\mathrm{wsp}(2N,\mathbb{R}) \supset \mathrm{sp}(2N,\mathbb{R}) \supset \mathrm{u}(N)$ bases.

This new application of VCS theory is of considerable interest not only because it paves the way to a further extension of the method, but also for its own sake. The $\mathrm{wsp}(2N,\mathbb{R})$ algebras indeed play an important role in various physical problems. On one hand, $\mathrm{wsp}(2N,\mathbb{R})$ provides a dynamical algebra for one of the most commonly used quantum mechanical systems, namely the N-dimensional harmonic oscillator (Wybourne 1974). On the other hand, $\mathrm{wsp}(6,\mathbb{R})$ finds an interesting application to the microscopic theory of nuclear collective motions (Rowe and Iachello 1983, Quesne 1987a, 1988).

In section 2, the definition of the wsp($2N$, ℝ) algebras and of their positive discrete series irreps is reviewed, and the existence of a Kantor decomposition with respect to an appropriate subalgebra is established. In section 3, the wsp($2N$, ℝ) VCS are introduced and shown to generalise the wsp(2, ℝ) coherent states recently studied by Beckers and Debergh (1989). The corresponding representation of the algebras is also obtained and its relation with a previously derived boson representation (Quesne 1988) is discussed. In section 4, the Kantor decomposition of wsp($2N$, ℝ) and K-matrix theory are used to construct orthonormal VCS bases symmetry-adapted to wsp($2N$, ℝ) $\supset$ sp($2N$, ℝ) $\supset$ u(N). Finally, in section 5, analytical expressions are obtained for the u(N)-reduced matrix elements of the wsp($2N$, ℝ) mod sp($2N$, ℝ) generators between two lowest-weight u(N) irrep states. They extend to arbitrary values of N some results previously obtained for $N=2$ (Quesne 1987b) and $N=3$ (Quesne 1988) by means of a very tedious raising operator technique.

2. The wsp(2N, ℝ) Lie algebras and their positive discrete series irreducible representations

The wsp($2N$, ℝ) algebra is defined as the semidirect sum of a Heisenberg-Weyl algebra w(N) and of a non-compact symplectic algebra sp($2N$, ℝ) (Quesne 1987a, 1988):

$$\mathrm{wsp}(2N,\mathbb{R}) = \mathrm{w}(N) \oplus \mathrm{sp}(2N,\mathbb{R}). \tag{2.1}$$

The generators of the former are denoted by I, $B_i^\dagger$, $B_i=(B_i^\dagger)^\dagger$, $i=1,\ldots,N$, and those of the latter by $E_{ij}=(E_{ji})^\dagger$, $D_{ij}^\dagger=D_{ji}^\dagger$, $D_{ij}=D_{ji}=(D_{ij}^\dagger)^\dagger$, $i,j=1,\ldots,N$.

The sp($2N$, ℝ) non-vanishing commutators are given by

$$\begin{aligned}
&[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \\
&[E_{ij}, D_{kl}^\dagger] = \delta_{jk}D_{il}^\dagger + \delta_{jl}D_{ik}^\dagger \qquad [E_{ij}, D_{kl}] = -\delta_{ik}D_{jl} - \delta_{il}D_{jk} \\
&[D_{ij}, D_{kl}^\dagger] = \delta_{ik}E_{lj} + \delta_{il}E_{kj} + \delta_{jk}E_{li} + \delta_{jl}E_{ki}
\end{aligned} \tag{2.2a}$$

showing that the operators E_{ij} span the maximal compact subalgebra u(N) of sp($2N$, ℝ). On the other hand, those of w(N)

$$[B_i, B_j^\dagger] = \delta_{ij}I \tag{2.2b}$$

express the bosonic character of the N pairs of operators $B_i^\dagger$, B_i, $i=1,\ldots,N$. Finally, the non-vanishing commutators of the sp($2N$, ℝ) generators with those of w(N),

$$\begin{aligned}
&[E_{ij}, B_k^\dagger] = \delta_{jk}B_i^\dagger \qquad [E_{ij}, B_k] = -\delta_{ik}B_j \\
&[D_{ij}, B_k^\dagger] = \delta_{ik}B_j + \delta_{jk}B_i \qquad [D_{ij}^\dagger, B_k] = -\delta_{ik}B_j^\dagger - \delta_{jk}B_i^\dagger
\end{aligned} \tag{2.2c}$$

indicate that w(N) is an invariant subalgebra of wsp($2N$, ℝ), and that $B_i^\dagger$ and B_i are vector operators with respect to sp($2N$, ℝ).

One can simultaneously diagonalise the sp($2N$, ℝ) weight generators $E_{11},\ldots,E_{NN}$, and the unit operator I. The subalgebra k, spanned by these $N+1$ operators, is the analogue of a Cartan subalgebra for a semisimple algebra. According to their transformation properties with respect to k, the set of wsp($2N$, ℝ) generators can be separated into three subsets of raising, weight, and lowering types as follows:

$$B_i^\dagger \quad D_{ij}^\dagger \quad E_{ij}\ (i<j) \qquad\quad E_{ii} \quad I \qquad\quad B_i \quad D_{ij} \quad E_{ij}\ (i>j) \tag{2.3}$$

where the subsets are separated by wider gaps.

The maximal compact subalgebra of $\mathrm{g} = \mathrm{wsp}(2N, \mathbb{R})$ is the direct sum algebra $\mathrm{h} = \mathrm{u}(1) \oplus \mathrm{u}(N)$, where $\mathrm{u}(1)$ and $\mathrm{u}(N)$ are spanned by I and E_{ij}, respectively. It obviously contains the generalised Cartan subalgebra k. Moreover, $\mathrm{wsp}(2N, \mathbb{R})$ can be given a Kantor structure

$$\mathrm{g} = \mathrm{g}_{-2} \oplus \mathrm{g}_{-1} \oplus \mathrm{g}_0 \oplus \mathrm{g}_{+1} \oplus \mathrm{g}_{+2} \tag{2.4}$$

where

$$\mathrm{g}_0 = \mathrm{h} = \mathrm{u}(1) \oplus \mathrm{u}(N) = \mathrm{span}\{I, E_{ij};\ 1 \leqslant i, j \leqslant N\} \tag{2.5}$$

contains the generator $\mathcal{N}$ of a $\mathrm{u}(1)$ subalgebra giving the grading, i.e. such that

$$[\mathcal{N}, g_\rho] = \rho g_\rho \qquad -2 \leqslant \rho \leqslant 2. \tag{2.6}$$

In other words, the elements of g belonging to various subspaces g_ρ satisfy the commutation relations

$$[g_\rho, g_\sigma] \subset g_{\rho+\sigma} \qquad -2 \leqslant \rho, \sigma \leqslant 2 \tag{2.7}$$

where $g_{\rho+\sigma} = 0$ if $|\rho + \sigma| > 2$. By taking

$$\mathcal{N} = E_{ii} \tag{2.8}$$

where from now on we shall assume that there is a summation over dummy indices, we indeed obtain

$$\begin{aligned} &\mathrm{g}_{-2} = \mathrm{span}\{D_{ij};\ 1 \leqslant i, j \leqslant N\} \qquad &&\mathrm{g}_{-1} = \mathrm{span}\{B_i;\ 1 \leqslant i \leqslant N\} \\ &\mathrm{g}_{+1} = \mathrm{span}\{B_i^\dagger;\ 1 \leqslant i \leqslant N\} \qquad &&\mathrm{g}_{+2} = \mathrm{span}\{D_{ij}^\dagger;\ 1 \leqslant i, j \leqslant N\}. \end{aligned} \tag{2.9}$$

The subalgebra h is not a maximal subalgebra of g. It is indeed contained in

$$\mathrm{l} = \mathrm{u}(1) \oplus \mathrm{sp}(2N, \mathbb{R}) = \mathrm{span}\{I, E_{ij}, D_{ij}^\dagger, D_{ij};\ 1 \leqslant i, j \leqslant N\} \tag{2.10}$$

which can be decomposed as follows:

$$\mathrm{l} = \mathrm{g}_{-2} \oplus \mathrm{g}_0 \oplus \mathrm{g}_{+2} \tag{2.11}$$

and therefore has a Jordan structure. The subalgebra chain

$$\mathrm{g} = \mathrm{wsp}(2N, \mathbb{R}) \supset \mathrm{l} = \mathrm{u}(1) \oplus \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{h} = \mathrm{u}(1) \oplus \mathrm{u}(N) \tag{2.12}$$

may be replaced by

$$\mathrm{g} = \mathrm{wsp}(2N, \mathbb{R}) \supset \mathrm{l}' = \mathrm{sp}(2N, \mathbb{R}) \supset \mathrm{h}' = \mathrm{u}(N) \tag{2.13}$$

since the $\mathrm{u}(1)$ subalgebra of l and h, being spanned by the unit operator, only has trivial irreps which, for all purposes, may be omitted. The chain (2.13) will play a fundamental role in the VCS construction and the K-matrix technique to be detailed in the next two sections.

In the present paper we are interested in those Hermitian irreps of $\mathrm{wsp}(2N, \mathbb{R})$ (exponentiating to unitary irreps of the corresponding semidirect product group $\mathrm{WSp}(2N, \mathbb{R}) = \mathrm{W}(N) \otimes \mathrm{Sp}(2N, \mathbb{R})$), which are encountered in physical applications, namely ladder irreps with a lowest-weight state (Quesne 1987a, 1988).

To build their carrier space, one starts from the basis states $|\{\Omega\}\alpha\rangle$ of a $\mathrm{u}(N)$ irrep $\{\Omega\} \equiv \{\Omega_1 \Omega_2 \ldots \Omega_N\}$, which are annihilated by all the $\mathrm{wsp}(2N, \mathbb{R})$ lowering generators belonging to $\mathrm{g}_- = \mathrm{g}_{-1} \oplus \mathrm{g}_{-2}$:

$$B_i |\{\Omega\}\alpha\rangle = D_{ij} |\{\Omega\}\alpha\rangle = 0. \tag{2.14}$$

Here α labels the basis of $\{\Omega\}$, and $\Omega_i = \lambda_i + n/2$, where n is some positive integer and $\{\lambda_1\lambda_2 \dots \lambda_N\}$ is some partition into non-negative integers. The space of states $|\{\Omega\}\alpha\rangle$ will be referred to as the lowest-weight u(N) subspace. The U(N) group leaving this subspace invariant, the corresponding algebra u(N) will be referred to as the stability subalgebra of wsp($2N$, ℝ)†.

Application of the raising generators belonging to $g_+ = g_{+1} \oplus g_{+2}$, i.e. the operators $B_i^\dagger$ and $D_{ij}^\dagger$, then generates the remaining basis states of the wsp($2N$, ℝ) irrep, to be denoted by $\langle\langle\Omega\rangle\rangle$. Its lowest-weight state is that of the lowest-weight u(N) irrep,

$$|\{\Omega\}\rangle \equiv |\{\Omega\} lw\rangle. \tag{2.15}$$

In addition to (2.14), it satisfies the relations

$$\begin{aligned} E_{ij}|\{\Omega\}\rangle &= 0 \qquad i>j \\ E_{ii}|\{\Omega\}\rangle &= \Omega_{N+1-i}|\{\Omega\}\rangle. \end{aligned} \tag{2.16}$$

3. Vector coherent states of wsp(2N, ℝ)

Following the general prescription of Rowe *et al* (1988), the construction of wsp($2N$, ℝ) VCS is based on the complex extension of the Kantor decomposition (2.4):

$$g^c = g^c_{-2} \oplus g^c_{-1} \oplus g^c_0 \oplus g^c_{+1} \oplus g^c_{+2}. \tag{3.1}$$

From (2.9), an arbitrary vector $Z \in g^c_- = g^c_{-1} \oplus g^c_{-2}$ can be expanded as

$$Z = y_i B_i + \tfrac{1}{2} z_{ij} D_{ij} \qquad z_{ij} = z_{ji} \tag{3.2}$$

where the $N(N+3)/2$ complex variables y_i and z_{ij} can be regarded as coordinates for the complex extension of the factor space $G/H = \mathrm{WSp}(2N, \mathbb{R})/[\mathrm{U}(1) \oplus \mathrm{U}(N)]$.

The wsp($2N$, ℝ) VCS are then defined by

$$|y, z; \alpha\rangle = \exp(Z^\dagger)|\{\Omega\}\alpha\rangle = \exp(y_i^* B_i^\dagger + \tfrac{1}{2} z_{ij}^* D_{ij}^\dagger)|\{\Omega\}\alpha\rangle. \tag{3.3}$$

They are parametrised by the complex variables y_i, z_{ij}, and by the discrete index α labelling a basis of the lowest-weight u(N) subspace (thence the alternative denomination of partially coherent states used elsewhere (Deenen and Quesne 1984b)). In the special case of wsp(2, ℝ), the u(1) irreps being one-dimensional, the VCS (3.3) reduce to standard coherent states

$$|y, z\rangle = \exp(y^* B^\dagger + \tfrac{1}{2} z^* D^\dagger)|\{\Omega\}\rangle. \tag{3.4}$$

The latter were recently studied by Beckers and Debergh (1989) for the irrep $\langle\langle\Omega\rangle\rangle = \langle\langle\tfrac{1}{2}\rangle\rangle$.

In the general case, the VCS representation of an arbitrary state $|\Psi\rangle$, belonging to the irrep $\langle\langle\Omega\rangle\rangle$ carrier space, is given by

$$\Psi(y, z) = \langle y, z|\Psi\rangle = \sum_\alpha |\{\Omega\}\alpha\rangle \Psi_\alpha(y, z) \tag{3.5}$$

where

$$\begin{aligned} \Psi_\alpha(y, z) &= \langle y, z; \alpha|\Psi\rangle = \langle\{\Omega\}\alpha|\exp(Z)|\Psi\rangle \\ &= \langle\{\Omega\}\alpha|\exp(y_i B_i + \tfrac{1}{2} z_{ij} D_{ij})|\Psi\rangle. \end{aligned} \tag{3.6}$$

† Strictly speaking, the stability subalgebra of wsp($2N$, ℝ) is u(1)⊕u(N). As explained in the text, u(1) may be omitted.

The function $\Psi(y, z)$ is therefore holomorphic in the variables y_i and z_{ij} and takes vector values in the lowest-weight u(N) subspace.

The carrier space for the wsp($2N, \mathbb{R}$) VCS representation is defined as the Hilbert space of all such holomorphic vector-valued functions which are square integrable with respect to the VCS scalar product

$$\langle\Psi'|\Psi\rangle_{\mathrm{VCS}} = \int [\Psi'(y, z)]^* \Psi(y, z)\, \mathrm{d}\sigma(y, z) \tag{3.7}$$

where $\mathrm{d}\sigma(y, z)$ is the VCS measure. The latter will not be explicitly determined in the present paper. We shall instead prove the existence of the scalar product $\langle\Psi'|\Psi\rangle_{\mathrm{VCS}}$, and therefore of the corresponding measure $\mathrm{d}\sigma(y, z)$, by specifying an orthonormal basis with respect to this scalar product. Such a construction will be carried out in the next section by the K-matrix technique.

The VCS representation of an arbitrary operator X acting in the carrier space of $\langle\langle\Omega\rangle\rangle$ is defined by

$$\begin{aligned}\Gamma(X)\Psi(y, z) &= \sum_\alpha |\{\Omega\}\alpha\rangle\langle y, z; \alpha|X|\Psi\rangle \\ &= \sum_\alpha |\{\Omega\}\alpha\rangle\langle\{\Omega\}\alpha|\exp(Z)X|\Psi\rangle.\end{aligned} \tag{3.8}$$

By applying the Baker–Campbell–Hausdorff formula

$$\exp(Z)X\exp(-Z) = X + \sum_{m=1}^{\infty} (m!)^{-1}[Z, [Z, \ldots, [Z, X]\ldots]]_m \tag{3.9}$$

the VCS representation of the wsp($2N, \mathbb{R}$) algebra is readily obtained:

$$\begin{aligned}&\Gamma(D_{ij}) = \nabla_{ij} \qquad \Gamma(B_i) = \partial_i \\ &\Gamma(E_{ij}) = \mathbb{E}_{ij} + \mathscr{E}_{ij}^{(y)} + \mathscr{E}_{ij}^{(z)} \qquad \mathscr{E}_{ij}^{(y)} = y_i\partial_j \qquad \mathscr{E}_{ij}^{(z)} = z_{ik}\nabla_{kj} \\ &\Gamma(B_i^\dagger) = z_{ij}\partial_j + y_i \\ &\Gamma(D_{ij}^\dagger) = z_{ik}\mathbb{E}_{jk} + z_{jk}\mathbb{E}_{ik} + (y_i z_{jk} + y_j z_{ik})\partial_k + z_{ik}z_{jl}\nabla_{kl} + y_i y_j.\end{aligned} \tag{3.10}$$

Here ∂_i and ∇_{ij} are differential operators defined by

$$\partial_i = \partial/\partial y_i \qquad \nabla_{ij} = (1 + \delta_{ij})\partial/\partial z_{ij} \tag{3.11}$$

and $\mathbb{E}_{ij}$ is the intrinsic representation of E_{ij}, i.e. the representation carried by the lowest-weight u(N) subspace

$$\mathbb{E}_{ij}\Psi(y, z) = \sum_\alpha E_{ij}|\{\Omega\}\alpha\rangle\Psi_\alpha(y, z). \tag{3.12}$$

At this point, it is worth noting that equation (3.3) is not the only possible definition of VCS for wsp($2N, \mathbb{R}$). Since the lowering generators B_i and D_{ij} commute with one another, they can be diagonalised simultaneously. As shown in appendix 1, their common eigenstates can also be chosen as VCS (Deenen and Quesne 1984b), thereby generalising the coherent states first considered for su(1, 1) $\simeq$ so(2, 1) $\simeq$ sp($2, \mathbb{R}$) by Barut and Girardello (1971), and later extended to sp($2N, \mathbb{R}$) by Deenen and Quesne (1984a). Such alternative VCS lead to another wsp($2N, \mathbb{R}$) VCS representation $\tilde{\Gamma}$, equivalent to Γ, and directly related to a wsp($2N, \mathbb{R}$) boson representation introduced recently (Quesne 1988).

4. Construction of an orthonormal wsp(2N, $\mathbb{R}$) ⊃ sp(2N, $\mathbb{R}$) ⊃ u(N) basis

Following Rowe *et al* (1988), we start by defining a basis of states which is orthonormal with respect to a Bargmann scalar product (Bargmann 1961)

$$\langle\Psi'|\Psi\rangle = \int [\Psi'(y, z)]^* \Psi(y, z)\, \mathrm{d}\mu(y, z) \tag{4.1}$$

where

$$\mathrm{d}\mu(y, z) = \left(\prod_i \mathrm{d}\mu(y_i)\right)\left(\prod_{i\leq j} \mathrm{d}\mu[(1+\delta_{ij})^{-1/2} z_{ij}]\right) \tag{4.2}$$

$$\mathrm{d}\mu(z) = \pi^{-1} \exp(-zz^*)\, \mathrm{d}\,\mathrm{Re}\, z\, \mathrm{d}\,\mathrm{Im}\, z. \tag{4.3}$$

For such a purpose, we note that the variables y_i and z_{ij} transform under u(N) in the same way as the $\mathrm{g}_+^{\mathrm{c}}$ basis elements $B_i^\dagger$ and $D_{ij}^\dagger$,

$$[\Gamma(E_{ij}), y_k] = \delta_{jk} y_i \qquad [\Gamma(E_{ij}), z_{kl}] = \delta_{jk} z_{il} + \delta_{jl} z_{ik} \tag{4.4}$$

i.e. as the components of $\{1\dot{0}\}$ and $\{2\dot{0}\}$ irreducible tensors, respectively (here a dot over a numeral implies that this numeral is repeated as often as necessary). We may therefore construct two sets of polynomials $P_\beta^{\{\nu\}}(z)$ and $Q_\gamma^{\{l\dot{0}\}}(y)$, transforming as the components of tensors of rank $\{\nu\}$ and $\{l\dot{0}\}$ under u(N). Here $\{\nu\} = \{\nu_1\nu_2 \ldots \nu_N\}$ denotes a partition into non-negative even integers, $\{l\dot{0}\}$ represents a symmetric u(N) irrep, and β, γ label the rows of $\{\nu\}$ and $\{l\dot{0}\}$ respectively. These polynomials span the spaces of polynomials in z_{ij} and y_i, and they may be chosen orthonormal with respect to the Bargmann measure so that

$$\langle P_{\beta'}^{\{\nu'\}}|P_\beta^{\{\nu\}}\rangle = \delta_{\{\nu'\},\{\nu\}}\delta_{\beta',\beta} \tag{4.5}$$

and

$$\langle Q_{\gamma'}^{\{l'\dot{0}\}}|Q_\gamma^{\{l\dot{0}\}}\rangle = \delta_{l',l}\delta_{\gamma',\gamma}. \tag{4.6}$$

Their highest-weight components are given by (Le Blanc and Rowe 1987)

$$P_{\mathrm{hw}}^{\{\nu\}}(z) = \mathcal{M}(\{\nu\})(z_{11})^{(\nu_1-\nu_2)/2}(z_{12,12})^{(\nu_2-\nu_3)/2} \ldots (z_{1\ldots N,1\ldots N})^{\nu_N/2} \tag{4.7}$$

$$Q_{\mathrm{hw}}^{\{l\dot{0}\}}(y) = (l!)^{-1/2}(y_1)^l \tag{4.8}$$

where

$$\mathcal{M}(\{\nu\}) = \left[\left(\prod_{i<j}^{N} (\nu_i - \nu_j + j - i)!![(\nu_i - \nu_j + j - i - 1)!!]^{-1}\right)\left(\prod_{i=1}^{N} [(\nu_i + N - i)!!]^{-1}\right)\right]^{1/2} \tag{4.9}$$

and $z_{1\ldots r,1\ldots r}$ denotes the determinant of order r formed from the first r rows and r columns of the $N\times N$ matrix $z = \|z_{ij}\|$.

We now observe that the VCS representation of u(N) is compatible with the Bargmann scalar product (4.1), which means that the Hermitian adjoint $\Gamma^\dagger(E_{ij})$ of $\Gamma(E_{ij})$ with respect to this scalar product satisfies the relation

$$\Gamma^\dagger(E_{ij}) = \Gamma((E_{ij})^\dagger) = \Gamma(E_{ji}) \tag{4.10}$$

as can be easily checked from (3.10). Hence it is possible to construct an orthonormal Bargmann basis of holomorphic vector-valued functions which reduces the stability subalgebra u(N)

$$\langle y, z|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\mu\rangle = [P^{\{\nu\}}(z)\times[Q^{\{l\dot{0}\}}(y)\times|\{\Omega\}\rangle]^{\{\omega\}}]_\mu^{\rho\{h\}}. \tag{4.11}$$

Such functions are obtained by coupling first the basis states $|\{\Omega\}\alpha\rangle$ of the lowest-weight $\mathrm{u}(N)$ subspace to the polynomials $Q_\gamma^{\{l\dot{0}\}}(y)$, then the resulting states to the polynomials $P_\beta^{\{\nu\}}(z)$ (note that in the present paper all couplings are assumed to be ordered sequentially from right to left). They are characterised by a given $\mathrm{u}(N)$ irrep $\{h\}$, a given row μ, and a set of multiplicity labels. The latter are the ranks $\{l\dot{0}\}$ and $\{\nu\}$ of the polynomials in y and z, the intermediate $\mathrm{u}(N)$ irrep $\{\omega\}$, and the multiplicity label ρ distinguishing between repeated irreps $\{h\}$ in $\{\omega\}\times\{\nu\}$. As a matter of fact, $\{l\dot{0}\}$ is redundant since l is fixed by $\{\Omega\}$ and $\{\omega\}$ as $l=\Sigma_i(\omega_i-\Omega_i)$.

We next seek a transformation K mapping the orthonormal Bargmann basis onto an orthonormal VCS basis

$$\langle y, z|\phi(\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\mu)\rangle = \langle y, z|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\mu\rangle. \tag{4.12}$$

The inverse transformation K^{-1} maps the $\mathrm{wsp}(2N,\mathbb{R})$ VCS representation Γ, which is unitary with respect to the VCS scalar product, onto an equivalent representation γ,

$$\gamma(X)=K^{-1}\Gamma(X)K \tag{4.13}$$

which is unitary with respect to the Bargmann scalar product:

$$\gamma(X^\dagger)=\gamma^\dagger(X)=K^\dagger\Gamma^\dagger(X)(K^{-1})^\dagger. \tag{4.14}$$

From (4.10), it results that K may be chosen so that the VCS and Bargmann representations of $\mathrm{wsp}(2N,\mathbb{R})$ are identical when restricted to the stability subalgebra $\mathrm{u}(N)$, i.e.

$$\Gamma(E_{ij})=\gamma(E_{ij}). \tag{4.15}$$

Thus K is diagonal in the $\mathrm{u}(N)$ representation labels $\{h\}$, and independent of μ:

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\nu'\}\rho'\{h'\}\mu'|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\mu\rangle\\&\quad=\delta_{\{h'\},\{h\}}\delta_{\mu',\mu}\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\nu'\}\rho'\{h\}|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\nu\}\rho\{h\}\rangle.\end{aligned} \tag{4.16}$$

Since the construction of orthonormal $\mathrm{sp}(2N,\mathbb{R})\supset\mathrm{u}(N)$ bases has been extensively studied elsewhere (Rowe 1984, Rowe *et al* 1984, 1985b, Deenen and Quesne 1984b, 1985), considerable simplification arises if we require the K operator to give VCS basis states reducing the subalgebra chain (2.13), and hence classified by the following labels

$$\begin{array}{ccccc}\mathrm{wsp}(2N,\mathbb{R}) & \supset & \mathrm{sp}(2n,\mathbb{R}) & \supset & \mathrm{u}(N)\\ \langle\!\langle\Omega\rangle\!\rangle & \{l\dot{0}\} & \langle\omega\rangle & \{\nu\}\rho & \{h\}\end{array} \tag{4.17}$$

where $\langle\omega\rangle$, in particular, characterises an $\mathrm{sp}(2N,\mathbb{R})$ irrep. We may then restrict the construction of an orthonormal VCS basis to that of a basis of lowest-weight $\mathrm{u}(N)$ irrep states

$$\langle y, z|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\mu\rangle. \tag{4.18}$$

From (3.10), it follows that the states (4.18) satisfy the relation

$$\Gamma(D_{ij})\langle y, z|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\}\omega\}\mu\rangle=\nabla_{ij}\langle y, z|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\mu\rangle=0. \tag{4.19}$$

Hence they must consist of a superposition of z-independent Bargmann basis states

$$\langle y|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\mu\rangle=[Q^{\{l\dot{0}\}}(y)\times|\{\Omega\}\rangle]_\mu^{\{\omega\}}. \tag{4.20}$$

This implies that

$$\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\nu'\}\rho'\{\omega\}|K|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle=\delta_{l',l}\delta_{\langle\omega'\rangle,\langle\omega\rangle}\delta_{\{\nu'\},\{\dot{0}\}}\mathscr{K}(\{\omega\}) \tag{4.21}$$

where $\mathcal{K}(\{\omega\})$ is a 1×1 submatrix of the full K matrix. Moreover, the VCS states (4.18) can be defined uniquely by choosing $\mathcal{K}(\{\omega\})$ real and normalised in such a way that $\mathcal{K}(\{\Omega\})=1$.

Arguments entirely similar to those of Rowe *et al* (1988), and based on the Hermiticity properties of the wsp($2N,\mathbb{R}$) generators, as well as on equations (4.13) and (4.14), easily lead to the following recursion relation for $\mathcal{K}(\{\omega\})^2$:

$$\begin{aligned}\mathcal{K}(\{\omega'\})^2&\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|[\Gamma^{(0)}(B)]^\dagger\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\\ &=\mathcal{K}(\{\omega\})^2\Bigg\{\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(B^\dagger)\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\\ &\quad-\sum_{l'',\{\omega''\}<\{\omega'\},\rho'',\{\nu''\}}\frac{\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma^{(0)}(D^\dagger)\|\langle\!\langle\Omega\rangle\!\rangle\{l''\dot{0}\}\langle\omega''\rangle\{\dot{0}\}\{\omega''\}\rangle}{\langle\langle\!\langle\Omega\rangle\!\rangle\{l''\dot{0}\}\langle\omega''\rangle\{\nu''\}\rho''\{\omega'\}\|\Gamma^{(1)}(D^\dagger)\|\langle\!\langle\Omega\rangle\!\rangle\{l''\dot{0}\}\langle\omega''\rangle\{\dot{0}\}\{\omega''\}\rangle}\\ &\quad\times\langle\langle\!\langle\Omega\rangle\!\rangle\{l''\dot{0}\}\langle\omega''\rangle\{\nu''\}\rho''\{\omega'\}\|\Gamma^{(1)}(B^\dagger)\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\Bigg\}\end{aligned}\tag{4.22}$$

where $\Gamma^{(0)}(X)$ denotes the z- and ∇-independent component of $\Gamma(X)$, $\Gamma^{(1)}(X)$ its component linear in z and ∇-independent, and all matrix elements are reduced with respect to u(N).

From (3.10), it follows that

$$[\Gamma^{(0)}(B_i)]^\dagger=\Gamma^{(0)}(B_i^\dagger)=y_i \tag{4.23a}$$

$$\Gamma^{(0)}(D_{ij}^\dagger)=y_iy_j \tag{4.23b}$$

$$\Gamma^{(1)}(B_i^\dagger)=z_{ij}\partial_j \tag{4.23c}$$

$$\Gamma^{(1)}(D_{ij}^\dagger)=z_{ik}(\mathbb{E}_{jk}+\mathscr{E}_{jk}^{(y)})+z_{jk}(\mathbb{E}_{ik}+\mathscr{E}_{ik}^{(y)})=[\Lambda, z_{ij}] \tag{4.23d}$$

where

$$\Lambda=\tfrac{1}{2}[\gamma(E_{ij})\gamma(E_{ji})-\mathscr{E}_{ij}^{(z)}\mathscr{E}_{ji}^{(z)}] \tag{4.24}$$

is a combination of Casimir operators. Furthermore, as a consequence of the u(N) tensorial properties of y_i and z_{ij}, the only non-vanishing values of the reduced matrix element on the left-hand side of (4.22) correspond to $l'=l+1$, and $\{\omega'\}=\{\omega+\Delta^{(1)}(i)\}$, $i=1,\ldots,N$, while the summation on the right-hand side runs over $\{\omega''\}=\{\omega-\Delta^{(1)}(j)\}$, $j=1,\ldots,N$, with l'', $\{\nu''\}$ fixed by $l''=l-1$, $\{\nu''\}=\{2\dot{0}\}$, and ρ'' not needed. Here $\Delta^{(1)}(k)$ denotes a row vector of dimension N with vanishing entries everywhere except for the component k, which has value unity.

As shown in appendix 2, all the reduced matrix elements appearing in (4.22) can be readily evaluated by using standard u(N) recoupling techniques. By taking these results into account, the recursion relation becomes

$$\begin{aligned}\mathcal{K}(\{\omega+\Delta^{(1)}(i)\})^2=\Bigg[1-\sum_j(\omega_i+\omega_j-i-j)^{-1}\bigg(\prod_k(\omega_j-\Omega_k+k-j)\bigg)\\ \times\bigg(\prod_{k\neq j}(\omega_j-\omega_k+k-j)\bigg)^{-1}\Bigg]\mathcal{K}(\{\omega\})^2.\end{aligned}\tag{4.25}$$

The sum over j on the right-hand side of (4.25) is performed in appendix 2, so that the recursion relation finally takes the simple form

$$\mathcal{K}(\{\omega+\Delta^{(1)}(i)\})^2=\bigg(\prod_j(\omega_i+\Omega_j-i-j)(\omega_i+\omega_j-i-j)^{-1}\bigg)\mathcal{K}(\{\omega\})^2. \tag{4.26}$$

Its solution is given by

$$\mathcal{K}(\{\omega\})^2=\prod_{i=1}^{N}\left[(\omega_i+\Omega_i-2i-1)![(2\Omega_i-2i-1)!!(2\omega_i-2i-2)!!]^{-1}\right.$$
$$\times\left(\prod_{j=1}^{i-1}(\omega_i+\Omega_j-i-j-1)!(\Omega_i+\omega_j-i-j-1)!\right.$$
$$\left.\left.\times[(\Omega_i+\Omega_j-i-j-1)!(\omega_i+\omega_j-i-j-1)!]^{-1}\right)\right] \qquad (4.27)$$

thus completing the construction of an orthonormal VCS basis of lowest-weight $\mathrm{u}(N)$ irrep states.

Observe that such states are in one-to-one correspondence with the corresponding Bargmann states (4.20) since none of the latter is annihilated by the K operator. This is due to the non-compact nature of $\mathrm{wsp}(2N,\mathbb{R})$. From VCS theory, we have therefore found that the only $\mathrm{sp}(2N,\mathbb{R})$ subirreps $\langle\omega\rangle$ appearing in the $\mathrm{wsp}(2N,\mathbb{R})$ irrep $\langle\!\langle\Omega\rangle\!\rangle$ are those which can be obtained, as in equation (4.20), by $\mathrm{u}(N)$ coupling a lowest-weight $\mathrm{u}(N)$ basis state $|\{\Omega\}\alpha\rangle$ to a polynomial $Q_\gamma^{\{l\dot{0}\}}(y)$, whose degree l runs over all non-negative integers. This result is consistent with, and in fact directly proves, the branching rule (Quesne 1988)

$$\mathrm{wsp}(2N,\mathbb{R})\downarrow\mathrm{sp}(2N,\mathbb{R})\colon\langle\!\langle\Omega\rangle\!\rangle\downarrow\sum_{\omega_1=\Omega_1}^{\infty}\sum_{\omega_2=\Omega_2}^{\Omega_1}\cdots\sum_{\omega_N=\Omega_N}^{\Omega_{N-1}}\oplus\langle\omega\rangle \qquad (4.28)$$

where each $\langle\omega\rangle$ is multiplicity free.

5. Matrix elements of the wsp(2N, ℝ) mod sp(2N, ℝ) generators

Having determined the $\mathcal{K}(\{\omega\})$ submatrices, we can now easily calculate the $\mathrm{u}(N)$-reduced matrix elements of the $\mathrm{wsp}(2N,\mathbb{R})\bmod\mathrm{sp}(2N,\mathbb{R})$ generators provided we restrict ourselves to lowest-weight $\mathrm{u}(N)$ irrep basis states. For such purpose, we shall use the Bargmann states (4.20) and the γ representation (4.13) of the $\mathrm{wsp}(2N,\mathbb{R})$ algebra.

Since the operators B_i commute with the $\mathrm{sp}(2N,\mathbb{R})$ lowering generators D_{ij}, they can only lower states of a lowest-weight $\mathrm{u}(N)$ irrep to other lowest-weight irreps. Hence taking $\mathrm{u}(N)$ reduced matrix elements of (4.13) between two states of the type (4.20), we obtain

$$\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\mathcal{K}(\{\omega'\})\gamma(B)\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$
$$=\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\Gamma(B)\mathcal{K}(\{\omega\})\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle. \qquad (5.1)$$

Combining this result with equation (3.10) leads to the relation

$$\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(B)\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$
$$=(\mathcal{K}(\{\omega\})/\mathcal{K}(\{\omega'\}))\langle\langle\!\langle\Omega\rangle\!\rangle\}l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\partial\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle \qquad (5.2)$$

from which the reduced matrix elements of $\gamma(B^+)$ can be obtained through Hermitian conjugation as

$$\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|\gamma(B^+)\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$
$$=(\mathcal{K}(\{\omega'\})/\mathcal{K}(\{\omega\}))\langle\langle\!\langle\Omega\rangle\!\rangle\{l'\dot{0}\}\langle\omega'\rangle\{\dot{0}\}\{\omega'\}\|y\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle. \qquad (5.3)$$

From (4.26), (A2.3) and (A2.12), one can easily derive the following explicit expressions for the non-vanishing u(N) reduced matrix elements of $\gamma(B^\dagger)$ and $\gamma(B)$:

$$\langle\langle\Omega\rangle\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\gamma(B^\dagger)\|\langle\langle\Omega\rangle\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$

$$=\left(\prod_j(\omega_i+\Omega_j-i-j)(\omega_i-\Omega_j+j-i+1)\right.$$
$$\left.\times[(\omega_i+\omega_j-i-j)(\omega_i-\omega_j+j-i+1)]^{-1}\right)^{1/2} \tag{5.4}$$

$$\langle\langle\Omega\rangle\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(i)\}\|\gamma(B)\|\langle\langle\Omega\rangle\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle$$

$$=(-1)^{i-1}\left(-\prod_j(\omega_i+\Omega_j-i-j-1)(\omega_j-\Omega_j+j-i)\right.$$
$$\left.\times[(\omega_i+\omega_j-i-j-1-\delta_{ij})(\omega_i-\omega_j+j-i-1)]^{-1}\right)^{1/2}. \tag{5.5}$$

They extend to arbitrary N values the results given for $N=2$ in equation (30) of Quesne (1987b) and for $N=3$ in equations (3.56) and (3.57) of Quesne (1988). Note that the sign discrepancy between the latter and equations (5.4) and (5.5) is due to a different phase convention for the u(3) generator matrix elements. Quesne (1988) indeed uses Hecht's phase choice (Hecht 1965), whereas Gel'fand and Tseitlin's convention is used throughout the present paper (Gel'fand and Tseitlin 1950).

As a final point, note that if the sp($2N$, ℝ) ⊃ u(N) reduced Wigner coefficients for the coupling of a unitary irrep $\langle\omega\rangle$ with a non-unitary vector irrep were available, it would be possible to derive the sp($2N$, ℝ) triple reduced matrix elements of $\gamma(B^\dagger)$ and $\gamma(B)$, and consequently the matrix elements of these operators between two arbitrary basis states of $\langle\langle\Omega\rangle\rangle$. The determination of those reduced Wigner coefficients is, however, made difficult by the multiplicity labels present in an orthonormal sp($2N$, ℝ) ⊃ u(N) basis.

Appendix 1. An alternative vector coherent state representation of wsp(2N, ℝ)

The purpose of the present appendix is to review the definition and some properties of alternative VCS for wsp($2N$, ℝ).

Let us consider the common eigenstates $|y, z; \lambda)$ of the set of commuting non-Hermitian operators B_i, $1\leq i\leq N$, and D_{ij}, $1\leq i\leq j\leq N$, corresponding to some complex eigenvalues y_i^* and z_{ij}^*, respectively:

$$B_i|y, z;\lambda)=y_i^*|y, z;\lambda) \qquad D_{ij}|y, z;\lambda)=z_{ij}^*|y, z;\lambda). \tag{A1.1}$$

Here λ is an extra label characterising possibly independent solutions of (A1.1), and a round bracket notation is used to distinguish the eigenstates $|y, z; \lambda)$ from the VCS $|y, z; \alpha\rangle$, defined in (3.3).

It can be easily proved that (A1.1) does indeed have solutions and, moreover, that the number of independent solutions is equal to the dimension dim$\{\Omega\}$ of the lowest-weight u(N) irrep $\{\Omega\}$, so that the extra label λ may be identified with the index α labelling the basis states of the latter. Indeed let $\Phi^{(\lambda)}(y^*, z^*; y', z')$ represent the states $|y, z; \lambda)$ in the VCS representation defined in section 3. From (3.5), it follows that

$$\Phi^{(\lambda)}(y^*, z^*; y', z')=\sum_{\alpha'}|\{\Omega\}\alpha'\rangle\Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') \tag{A1.2}$$

where

$$\Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') = \langle y', z'; \alpha'|y, z; \lambda). \tag{A1.3}$$

By taking (3.10) into account, equation (A1.1) is replaced by the system of partial differential equations

$$\begin{aligned} \partial_i' \Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') &= y_i^* \Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') \\ \nabla_{ij}' \Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') &= z_{ij}^* \Phi_{\alpha'}^{(\lambda)}(y^*, z^*; y', z') \end{aligned} \tag{A1.4}$$

where $\partial_i' = \partial/\partial y_i'$, and $\nabla_{ij}' = \partial/\partial z_{ij}'$. Arguments entirely similar to those of Deenen and Quesne (1984b) and Quesne (1986b), that we shall not reproduce here, enable one to show that (A1.4) has exactly $\dim\{\Omega\}$ independent solutions, labelled by $\lambda = \alpha$

$$\Phi_{\alpha'}^{(\alpha)}(y^*, z^*; y', z') = \delta_{\alpha',\alpha} \exp(y_i' y_i^* + \tfrac{1}{2} z_{ij}' z_{ij}^*). \tag{A1.5}$$

Equations (A1.2) (with $\lambda = \alpha$) and (A1.5) uniquely define the states $|y, z; \alpha)$, which may be chosen as an alternative definition of VCS for $\mathrm{wsp}(2N, \mathbb{R})$.

In the corresponding VCS representation of $\mathrm{wsp}(2N, \mathbb{R})$, an arbitrary state $|\Psi\rangle$, belonging to the irrep $\langle\!\langle\Omega\rangle\!\rangle$ carrier space, is represented by the holomorphic vector-valued function

$$\tilde{\Psi}(y, z) = (y, z|\Psi\rangle = \sum_\alpha |\{\Omega\}\alpha\rangle \tilde{\Psi}_\alpha(y, z) \tag{A1.6}$$

where

$$\tilde{\Psi}_\alpha(y, z) = (y, z; \alpha|\Psi\rangle. \tag{A1.7}$$

An arbitrary operator X, acting in the same space, is represented by $\tilde{\Gamma}(X)$, whose definition is

$$\tilde{\Gamma}(X)\tilde{\Psi}(y, z) = \sum_\alpha |\{\Omega\}\alpha\rangle (y, z; \alpha|X|\Psi\rangle. \tag{A1.8}$$

It is straightforward to show that the two $\mathrm{wsp}(2N, \mathbb{R})$ VCS representations Γ and $\tilde{\Gamma}$ are related by the equation

$$\tilde{\Gamma}(X) = \Gamma^+(X^+) \tag{A1.9}$$

where, as in (4.10) and (4.14), Γ^+ is the Hermitian adjoint of Γ with respect to the Bargmann scalar product (4.1). For $X = B_i^+$, for instance, choosing for $|\Psi\rangle$ any VCS $|y', z'; \alpha'\rangle$, we indeed obtain

$$\tilde{\Gamma}(B_i^+)(y, z|y', z'; \alpha'\rangle = \sum_\alpha |\{\Omega\}\alpha\rangle \langle y', z'; \alpha'|B_i|y, z; \alpha)^*. \tag{A1.10}$$

Since, from (3.10) and (A1.4),

$$\langle y', z'; \alpha'|B_i|y, z; \alpha) = \partial_i' \langle y', z'; \alpha'|y, z; \alpha) = y_i^* \langle y', z'; \alpha'|y, z; \alpha) \tag{A1.11}$$

it follows from (A1.10) that

$$\tilde{\Gamma}(B_i^+) = y_i \tag{A1.12}$$

in agreement with (A1.9) and (3.10).

As a consequence of (4.13), (4.14) and (A1.9), we finally get the relations

$$\gamma(X) = K^+ \tilde{\Gamma}(X)(K^+)^{-1} \tag{A1.13}$$

$$\Gamma(X) = KK^+ \tilde{\Gamma}(X)(KK^+)^{-1} \tag{A1.14}$$

thus showing the equivalence of the three representations Γ, $\tilde{\Gamma}$ and γ of $\mathrm{wsp}(2N, \mathbb{R})$.

Note that by substituting in the usual way (Deenen and Quesne 1984b) boson creation and annihilation operators $a^\dagger_{ij} = a^\dagger_{ji}$, $b^\dagger_i$, $a_{ij} = (a^\dagger_{ij})^\dagger$, and $b_i = (b^\dagger_i)^\dagger$ for z_{ij}, y_i, ∇_{ij} and ∂_i, we directly recover the boson representation of wsp(2N, $\mathbb{R}$), recently obtained by Quesne (1988).

Appendix 2. Recursion relation for $\mathscr{K}(\{\omega\})^2$

The aim of the present appendix is to prove that the recursion relation (4.22) for $\mathscr{K}(\{\omega\})^2$ can be put into the simple form (4.25) or (4.26). For this purpose, we have to evaluate the various u(N) reduced matrix elements appearing in (4.22).

The reduced matrix elements of the rank $\{1\dot{0}\}$ irreducible tensor $Q^{\{1\dot{0}\}}(y) = y$ can be expressed as

$$\begin{aligned}\langle\langle\!\langle\Omega\rangle\!\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|y\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle \\ = U(\{\Omega\}\{l\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\{1\dot{0}\};\{\omega\}\{l+1\dot{0}\})\langle\{l+1\dot{0}\}\|y\|\{l\dot{0}\}\rangle \qquad \text{(A2.1)}\end{aligned}$$

where the U coefficient is a u(N) Racah coefficient in unitary form (Hecht *et al* 1981, 1987), which can be evaluated from equation (A10) of Le Blanc and Hecht (1987). As a consequence of (4.8), the reduced matrix element on the right-hand side of (A2.1) is simply given by

$$\langle\{l+1\dot{0}\}\|y\|\{l\dot{0}\}\rangle = \sqrt{l+1}. \qquad \text{(A2.2)}$$

Hence, equation (A2.1) becomes

$$\begin{aligned}\langle\langle\!\langle\Omega\rangle\!\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|y\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle \\ = \left(\prod_{k\neq i}(\omega_i-\omega_k+k-i+1)\right)^{-1/2}\left(\prod_k(\omega_i-\Omega_k+k-i+1)\right)^{1/2}. \qquad \text{(A2.3)}\end{aligned}$$

In accordance with (4.7), the highest-weight component of the rank $\{2\dot{0}\}$ irreducible tensor in (4.23*b*) is defined by

$$\Gamma^{(0)}(D^\dagger_{\rm hw}) = 2^{-1/2}\Gamma^{(0)}(D^\dagger_{11}) = 2^{-1/2}y_1^2 = Q^{\{2\dot{0}\}}_{\rm hw}(y). \qquad \text{(A2.4)}$$

With this normalisation, $\Gamma^{(0)}(D^\dagger)$ can be rewritten as

$$\Gamma^{(0)}(D^\dagger) = 2^{-1/2}[Q^{\{1\dot{0}\}}(y)\times Q^{\{1\dot{0}\}}(y)]^{\{2\dot{0}\}} \qquad \text{(A2.5)}$$

and its reduced matrix elements are given by†

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(0)}(D^\dagger) \\ &\qquad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle \\ &\quad= 2^{-1/2}\sum_{\{\bar{\omega}\}}U(\{1\dot{0}\}\{1\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\{\omega-\Delta^{(1)}(j)\};\{2\dot{0}\}\{\bar{\omega}\}) \\ &\qquad\times\langle\langle\!\langle\Omega\rangle\!\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\rangle\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|y\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\bar{\omega}\rangle\{\dot{0}\}\{\bar{\omega}\}\rangle \\ &\qquad\times\langle\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\bar{\omega}\rangle\{\dot{0}\}\{\bar{\omega}\}\|y\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle. \qquad \text{(A2.6)}\end{aligned}$$

† Note that the reduced matrix elements of $Q^{\{2\dot{0}\}}(y)$ cannot be directly evaluated without making use of the decomposition (A2.5) because, as far as the author knows, the required Racah coefficients cannot be found in the literature.

Here the summation runs over $\{\bar{\omega}\}=\{\omega+\Delta^{(1)}(i)-\Delta^{(1)}(j)\}$, $\{\omega\}$ whenever $i\neq j$, and over $\{\bar{\omega}\}=\{\omega\}$ whenever $i=j$, and the corresponding Racah coefficients can be calculated from equation (A9) of Le Blanc and Rowe (1987). After inserting (A2.3) into (A2.6), we obtain

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l+1\dot{0}\}\langle\omega+\Delta^{(1)}(i)\}\{\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(0)}(D^{\dagger})\\&\quad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle\\&\quad=\left[(1+\delta_{ij})\left(\prod_{k\neq i}(\omega_i-\omega_k+k-i+1+\delta_{kj})\right)\left(\prod_{k\neq j}(\omega_j-\omega_k+k-j)\right)\right]^{-1/2}\\&\quad\times\left(\prod_{k}(\omega_i-\Omega_k+k-i+1)(\omega_j-\Omega_k+k-j)\right)^{1/2}.\end{aligned}\tag{A2.7}$$

In coupled tensor form, equation (4.23c) can be rewritten as

$$\Gamma^{(1)}(B^{\dagger})=(-1)^{N-1}\sqrt{N+1}[P^{\{2\dot{0}\}}(z)\times Q^{\{\dot{0}-1\}}(\boldsymbol{\partial})]^{\{1\dot{0}\}}\tag{A2.8}$$

where $P^{\{2\dot{0}\}}(z)=z$ is a rank $\{2\dot{0}\}$ irreducible tensor, defined in accordance with equation (4.7), and $Q^{\{\dot{0}-1\}}(\boldsymbol{\partial})=\partial$ is a rank $\{\dot{0}-1\}$ irreducible tensor, which is Hermitian conjugate to $Q^{\{1\dot{0}\}}(y)=y$ and whose lowest-weight component is given by

$$Q^{\{\dot{0}-1\}}_{\mathrm{lw}}(\boldsymbol{\partial})=\partial_1.\tag{A2.9}$$

The u(N) Wigner coefficients required in (A2.8) are taken from Biedenharn and Louck (1968).

The reduced matrix elements of $\Gamma^{(1)}(B^{\dagger})$ can therefore be expressed as

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{2\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(1)}(B^{\dagger})\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\\&\quad=(-1)^{N-1}\sqrt{N+1}\,U(\{\omega\}\{\dot{0}-1\}\{\omega+\Delta^{(1)}(i)\}\{2\dot{0}\};\{\omega-\Delta^{(1)}(j)\}\{1\dot{0}\})\\&\quad\times\langle\{2\dot{0}\}\|z\|\{\dot{0}\}\rangle\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\|\partial\\&\quad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\end{aligned}\tag{A2.10}$$

where the reduced matrix element of z is given by (Le Blanc and Rowe 1987)

$$\langle\{2\dot{0}\}\|z\|\{\dot{0}\}\rangle=1\tag{A2.11}$$

the reduced matrix element of ∂ can be evaluated from that of y, given in (A2.3), as follows

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\|\partial\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\\&\quad=(-1)^{j-1}\left(\prod_{k\neq j}(\omega_j-\omega_k+k-j-1)\right)^{-1/2}\left(\prod_{k}(\omega_j-\Omega_k+k-j)\right)^{1/2}\end{aligned}\tag{A2.12}$$

and the Racah coefficient can be calculated from equation (A9) of Le Blanc and Hecht (1987) by using some symmetry properties of Racah coefficients (Hecht *et al* 1981). Equation (A2.10) then becomes

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{2\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(1)}(B^{\dagger})\|\langle\!\langle\Omega\rangle\!\rangle\{l\dot{0}\}\langle\omega\rangle\{\dot{0}\}\{\omega\}\rangle\\&\quad=\left[(\omega_i-\omega_j+j-i+1)\left(\prod_{k\neq j}(\omega_j-\omega_k+k-j)\right)\right]^{-1/2}\\&\quad\times\left[(\omega_i-\omega_j+j-i+2)\left(\prod_{k}(\omega_j-\Omega_k+k-j)\right)\right]^{1/2}.\end{aligned}\tag{A2.13}$$

From (4.23*d*), the reduced matrix elements of $\Gamma^{(1)}(D^{\dagger})$ can be written as

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{2\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(1)}(D^{\dagger})\\ &\qquad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle\\ &\qquad=[\lambda(\{2\dot{0}\},\{\omega+\Delta^{(1)}(i)\})-\lambda(\{\dot{0}\},\{\omega-\Delta^{(1)}(j)\})]\\ &\qquad\times\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{2\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|z\\ &\qquad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle\end{aligned}\tag{A2.14}$$

where

$$\lambda(\{\nu\},\{h\})=\frac{1}{2}\sum_i[h_i(h_i+N-2i+1)-\nu_i(\nu_i+N-2i+1)]\tag{A2.15}$$

is the eigenvalue of the operator Λ, defined in (4.24), corresponding to the state (4.11). Since, by using (A2.11), the reduced matrix element of z can be shown to be equal to 1, we find

$$\begin{aligned}&\langle\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{2\dot{0}\}\{\omega+\Delta^{(1)}(i)\}\|\Gamma^{(1)}(D^{\dagger})\\ &\qquad\times\|\langle\!\langle\Omega\rangle\!\rangle\{l-1\dot{0}\}\langle\omega-\Delta^{(1)}(j)\rangle\{\dot{0}\}\{\omega-\Delta^{(1)}(j)\}\rangle=\omega_i+\omega_j-i-j.\end{aligned}\tag{A2.16}$$

By inserting (A2.3), (A2.7), (A2.13) and (A2.16) into (4.22), it is now a simple matter to derive equation (4.25). It then remains to prove that the latter is equivalent to equation (4.26).

The sum over j on the right-hand side of (4.25) can be performed by using complex function residue theory (Le Blanc and Rowe 1987). In terms of the partial hooks

$$p_{jN}=\omega_j+N-j\qquad P_{jN}=\Omega_j+N-j\tag{A2.17}$$

it can indeed be rewritten as

$$\sum_j\left[\frac{p_{jN}-P_{jN}}{p_{iN}+p_{jN}-2N}\prod_{k\neq j}\left(\frac{p_{jN}-P_{kN}}{p_{jN}-p_{kN}}\right)\right]=\sum_j\operatorname{Res}f(p_{jN})\tag{A2.18}$$

where $\operatorname{Res}f(p_{jN})$ is the residue of the complex function

$$f(z)=\frac{1}{z+p_{iN}-2N}\prod_k\left(\frac{z-P_{kN}}{z-p_{kN}}\right)\tag{A2.19}$$

at the point p_{jN}. Since

$$\begin{aligned}\sum_j\operatorname{Res}f(p_{jN})&=-\operatorname{Res}f(2N-p_{iN})-\operatorname{Res}f(\infty)\\ &=-\prod_k\left(\frac{p_{iN}+P_{kN}-2N}{p_{iN}+p_{kN}-2N}\right)+1\end{aligned}\tag{A2.20}$$

equation (4.26) is finally obtained.

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